<u>**Partial Fraction Exercise**</u>

1. Resolve the expression $\dfrac{x-2}{(x^2+1)(x-1)^2}$ into simplest partial fractions.

2. Let g(x) be a quadratic polynomial and a, b, c distinct constants.

If $\dfrac{g(x)}{(x-a)(x-b)(x-c)} = \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$ where A, B, C are constants, express A in terms of a, b, c and g(a).

Hence or otherwise resolve $\dfrac{x^2}{(x-1)(x-2)(x-3)}$ into partial fractions.

3. Prove that, if a, b, c are unequal,

$$\frac{a^2}{bc(a-b)(a-c)} + \frac{b^2}{ca(b-c)(b-a)} + \frac{c^2}{ab(c-a)(c-b)} \equiv \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}$$

4. Find the coefficients of A, B, C, D so that the following equation may be true for all values of x,

$$\frac{1+x^2}{(x^2-4)(2x^2+5)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{2x^2+5}$$

5. Establish the identity: $\dfrac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \dfrac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \dfrac{c^2(x-a)(x-b)}{(c-a)(c-b)} \equiv x^2$

6. Resolve into partial fractions:

(a) $\dfrac{2x^3-8x^2+x+6}{(x-3)^4}$ **(b)** $\dfrac{5x^2-4x+16}{(x-3)(x^2-x+1)^2}$

7. By expressing $(x-a)^4$ as $(x+a-2a)^4$, or otherwise, express $\dfrac{(x-a)^4}{(x+a)^4}$ into partial fractions.

8. If $y = \dfrac{a-b}{(x-a)(x-b)}$, express y and y^2 in partial fractions.

9. Express as a sum of partial fractions:

(a) $\dfrac{3x^2+4x+7}{x^2-3x+2}$ **(b)** $\dfrac{1+3x}{(1+2x)(1-x)^2}$ **(c)** $\dfrac{x^2}{1-x^4}$

(d) $\dfrac{9}{(1-2x)(1+x)^2}$ **(e)** $\dfrac{5-7x}{2x^3-x^2-2x+1}$ **(f)** $\dfrac{x^3+7x^2+9x-14}{(x+3)(4-x^2)}$

(g) $\dfrac{x}{(1+x^2)(1+x)^2}$

10. Express $\dfrac{1}{x(x+2)}$ in partial fractions. Hence find the sum of n terms of the series: $\dfrac{1}{1\times3} + \dfrac{1}{2\times4} + \dfrac{1}{3\times5} + \dots\dots$

11. Resolve $\dfrac{1}{x(x-2)(x-1)^{2n}}$ into partial fractions when n is a positive integer.

12. Express $\dfrac{2x+3}{x(x+1)(x+2)}$ into partial fractions.

13. Resolve $\dfrac{x}{(x+1)(x+2)(x+3)}$ into partial fractions. Hence find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)}$.

14. Using partial fractions, find the sum of:

(a) $\dfrac{1}{1\times2\times3}+\dfrac{1}{2\times3\times4}+\dfrac{1}{3\times4\times5}+\cdots+\dfrac{1}{n(n+1)(n+2)}$

(b) $\dfrac{4}{2\times3\times4}+\dfrac{7}{3\times4\times5}+\dfrac{10}{4\times5\times6}+\cdots+\dfrac{4n+1}{(n+1)(n+2)(n+3)}$

15. **(a)** Write down the formula for the sum of the coefficients in the expansion of $(1 + x)^m$, where m is a positive integer.

(b) Deduce $\dfrac{1}{1!(2n)!}+\dfrac{1}{2!(2n-1)!}+\dfrac{1}{3!(2n-2)!}+\ldots+\dfrac{1}{n!(n+1)!}=\dfrac{2^{2n-1}-1}{(2n+1)!}$

16. Prove that $\dfrac{(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)}=1+\displaystyle\sum_{r=1}^{n}\frac{(-1)^{n-r+1}(n+r)!}{(n-r)!r!(r-1)!(x+r)}$ and show that $\displaystyle\sum_{r=1}^{n}\frac{(-1)^{r+1}(n+r)!}{(r!)^2(n-r)!}=1-(-1)^n$.

17. Express $\dfrac{1}{x(x+1)\cdots(x+n)}$ in partial fractions. Hence, show that $\displaystyle\sum_{r=0}^{n}\frac{a^r}{x(x+1)\cdots(x+r)}=\sum_{s=0}^{n}\frac{(-1)^s A_s}{n!(n-s)!(x+s)}$

where the A_s are polynomials of degree n in a with integral coefficients.

18. Prove that $\dfrac{x}{x^5-1}$ when expressed in partial fractions is $\dfrac{1}{5}\cdot\dfrac{1}{x-1}+\dfrac{2}{5}\cdot\dfrac{x\cos\frac{4\pi}{5}-2\cos\frac{2\pi}{5}}{x^2-2x\cos\frac{2\pi}{5}+1}+\dfrac{2}{5}\cdot\dfrac{x\cos\frac{8\pi}{5}-2\cos\frac{4\pi}{5}}{x^2-2x\cos\frac{4\pi}{5}+1}$.

19. Express in partial fractions the function: $\dfrac{x^p}{(x+1)\cdots(x+n)}$ in the two cases **(a)** $1 \le p \le n-1$, **(b)** $p = n$.

Hence or otherwise prove that the expression $\dfrac{1^p}{1!(n-1)!}-\dfrac{2^p}{2!(n-2)!}+\cdots+\dfrac{(-1)^{n-2}(n-1)^p}{(n-1)!1!}+\dfrac{(-1)^{n-1}n^p}{n!}$

takes the value zero when $p = 0, 1, 2, \ldots, n-1$, and find its value when $p = n$.

20. Prove that if $(1 + x)^n = c_0 + c_1x + \ldots + c_nx^n$, then

$$\frac{c_0}{y}-\frac{c_1}{y+1}+\frac{c_2}{y+2}-\ldots+(-1)^n\frac{c_n}{y+n}=\frac{n!}{y(y+1)(y+2)\ldots(y+n)}$$ where y is not zero or a negative integer.

21. Given that $f(x) = (x - a_1)(x - a_2)\ldots(x - a_n)$ and $a_1, a_2, \ldots, a_n$ are unequal.

(a) Prove that $\dfrac{f'(x)}{f(x)}=\displaystyle\sum_{r=1}^{n}\frac{1}{x-a_r}$.

(b) Prove that $\dfrac{[f'(x)]^2-f(x)f''(x)}{[f(x)]^2}=\displaystyle\sum_{r=1}^{n}\frac{1}{(x-a_r)^2}$.

(c) If $\phi(x)$ is a polynomial of degree < n, prove that $\dfrac{\phi(x)}{f(x)}=\displaystyle\sum_{r=1}^{n}\frac{\phi(a_r)}{f'(a_r)}\frac{1}{x-a_r}$.

If $f(x) = (x - a_r)g_r(x)$, show that the above result can also be put in the form $\dfrac{\phi(x)}{f(x)}=\displaystyle\sum_{r=1}^{n}\frac{\phi(a_r)}{g_r(a_r)}\frac{1}{x-a_r}$.

22. Given that $f(x) = (x - a_0)(x - a_1) \ldots (x - a_n)$, where $a_0, a_1, \ldots, a_n$ are all distinct, and $\phi(x)$ is a polynomial of degree not greater than n + 1, show that $\sum_{r=0}^{n} \frac{\phi(a_r)}{f'(a_r)}$ is the coefficient of x^n in $\phi(x)$.

23. Given that for all x, $$\frac{ax^2 + bx + c}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}$$

find the condition that $A + B + C = 0$. Hence or otherwise, evaluate $\sum_{n=1}^{N} \frac{3n+1}{n(n+1)(n+2)}$.

24. **(a)** Find the values of A, B, C, D so that

$$\frac{3+x^2}{(1-x)^2(1+x^2)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C+Dx}{1+x^2}$$ becomes an identity.

(b) Prove that when $-1 < x < 1$, $\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \ldots = \frac{x}{1-x}$

25. Resolve the expression $\frac{x^6 - x^2 + 1}{(x-1)^3}$ into partial fraction.

26. Determine the constant A in the identity: $\left(\frac{x}{y}+1\right)\left(\sqrt{2} - \frac{x}{y}\right)\left(\sqrt{2} - \frac{x+2y}{x+y}\right) \equiv A\left(\sqrt{2} - \frac{x}{y}\right)^2$.

Using this identity, prove:

(i) that the irrational number $\sqrt{2}$ lies between the positive rational number $\frac{m}{n}$ and $\frac{m+2n}{m+n}$.

(ii) that $\sqrt{2}$ is closer to $\frac{m+2n}{m+n}$ than to $\frac{m}{n}$.

27. Let n be any positive integer, and a_r, b_r the coefficients of x^r in $(1 + x)^n$ and $(1 + x)^{n+2}$ respectively. Prove that:

(a) $b_{r+2} = a_r + 2a_{r+1} + a_{r+2}$ if $0 \le r \le n - 2$.

(b) $\frac{n!}{x(x+1)\ldots(x+n)} = \sum_{r=0}^{n} \frac{(-1)^r a_r}{x+r}$,

(c) $\frac{a_0}{x(x+1)(x+2)} - \frac{a_1}{(x+1)(x+2)(x+3)} + \ldots + \frac{(-1)^n a_n}{(x+n)(x+n+1)(x+n+2)} = \frac{(n+2)!}{2x(x+1)\ldots(x+n+2)}$

28. **(a)** Let A(x) be a polynomial of degree n in x, with real coefficients and n real roots $x_1, x_2, \ldots, x_n$.

Prove that $\sum_{i=1}^{n} \frac{1}{x-x_i} = \frac{A'(x)}{A(x)}$, where A'(x) is the derivative of A(x).

Hence or otherwise, prove that $\sum_{i=1}^{n} \frac{1}{(x-x_i)^2} = \frac{[A'(x)]^2 - A(x)A''(x)}{[A(x)]^2}$

(b) Resolve $\frac{2x-1}{(x-1)^2}$ into partial fraction.

(c) Let x_1, x_2, x_3, x_4 be roots of the polynomial

$B(x) = x^4 - 10x^2 + 1$. (You may assume that all the roots of B(x) are real.)

Using (a) and (b) or otherwise, evaluate the sum: $\sum_{i=1}^{4} \frac{2x_i - 1}{(x_i - 1)^2}$.

(For Q.29 – 34, the concept of Binomial series is needed.)

29. Resolve $\dfrac{7-8x}{(1-x)(2-x)}$ into partial fractions. Hence find the expansion of the function in ascending powers of x.

State for what range of values of x the expansion is valid and prove that, from the fourth terms onwards, the coefficients are all negative.

30. Resolve into partial fractions $\dfrac{1+x}{(1+2x)^2(1-x)}$.

For what range of values of x can this function be expanded as a series in ascending powers of x?

Write down the coefficient of x^n in this expansion.

31. Resolve into partial fractions $\dfrac{2}{(1-2x)^2(1+4x^2)}$ and hence obtain the coefficients of x^{4n} and x^{4n+1} in the expansion of this function in ascending powers of x.

State the range of values of x for which the expansion is valid.

32. Resolve into partial fractions $\dfrac{x^2+1}{(x-3)^2(x-2)}$ and hence obtain the coefficient of x^n in the expansion of this function in ascending powers of x.

For what range of values of x for which the expansion is valid?

33. Resolve into partial fractions $\dfrac{2+x^2}{(2-x)^2(4+x)}$ and expand the function in a series of ascending powers of x. Find the coefficient of x^n and state the range of values of x for which the expansion is valid.

34. Express $\dfrac{3x+4}{(x+1)(x+2)^2}$ in partial fractions. Hence obtain the expansion of the given expression in ascending powers of x as far as the term in x^3, stating the necessary restrictions on the value of x.